

# Some stochastic time-fractional diffusion equations with variable coefficients and time dependent noise

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## Abstract

We prove the existence and uniqueness of mild solution for the stochastic partial differential equation

$$(\partial^\alpha - B)u(t, x) = u(t, x) \cdot \dot{W}(t, x),$$

where

$$\alpha \in (1/2, 1) \cup (1, 2);$$

$B$  is an uniform elliptic operator with variable coefficients and  $\dot{W}$  is a Gaussian noise general in time with space covariance given by fractional, Riesz and Bessel kernel.

**Keywords:** Gaussian noisy environment, time fractional order spde, Fox H-functions, mild solutions, uniform elliptic operator, chaos expansion, Riesz kernel, Bessel kernel.

## 1 Introduction

In this article we prove the existence and uniqueness of the mild solution of the equation

$$\begin{cases} (\partial^\alpha - B)u(t, x) = u(t, x)\dot{W}(t, x), & t \in (0, T], x \in \mathbb{R}^d, \\ \left. \frac{\partial^k}{\partial t^k} u(t, x) \right|_{t=0} = u_k(x), & 0 \leq k \leq [\alpha] - 1, x \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

with any fixed  $T \in \mathbb{R}^+$ ,  $\alpha \in (1/2, 1) \cup (1, 2)$ , where  $[\alpha]$  is the smallest integer not less than  $\alpha$ . Here we assume

- $u_0(x)$  is bounded continuously differentiable. Its first order derivative bounded and Hölder continuous. The Hölder exponent  $\gamma > \frac{2-\alpha}{\alpha}$
- $u_1(x)$  is bounded continuous function (locally hölder continuous if  $d > 1$ )

In this equation,  $\dot{W}$  is a zero mean Gaussian noise with the following covariance structure

$$\mathbb{E}(\dot{W}(t, x)\dot{W}(s, y)) = \lambda(t - s)\Lambda(x - y),$$

where  $\lambda(\cdot)$  is nonnegative definite and locally intergrable and  $\Lambda(\cdot)$  is one of the following situations:

- (i) Fractional kernel.  $\Lambda(x) := \prod_{i=1}^d 2H_i(2H_i - 1)|x_i|^{2H_i-1}$ ,  $x \in \mathbb{R}^d$  and  $1/2 < H_i < 1$ .
- (ii) Reisz kernel.  $\Lambda(x) := C_{\alpha,d}|x|^{-\kappa}$ ,  $x \in \mathbb{R}^d$  and  $0 < \kappa < d$  and  $C_{\alpha,d} = \Gamma(\frac{\kappa}{2})2^{-\alpha}\pi^{-d/2}/\Gamma(\frac{\alpha}{2})$ .
- (iii) Bessel kernel.  $\Lambda(x) := C_{\alpha} \int_0^{\infty} \omega^{-\frac{\kappa}{2}-1} e^{-\omega} e^{\frac{-|x|^2}{4\omega}} d\omega$ ,  $x \in \mathbb{R}^d$ ,  $0 < \kappa < d$ , and  $C_{\alpha} = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$ .

$$B := \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(x) \frac{\partial}{\partial x_j} + c(x)$$

is uniformly elliptic. Namely it satisfies the following conditions:

- (i)  $a_{ij}(x)$ ,  $b_j(x)$  and  $c(x)$  are bounded Hölder continuous functions on  $\mathbb{R}^d$
- (ii)  $\exists a_0 > 0$ , such that  $\forall x, \xi \in \mathbb{R}^d$ ,

$$\sum_{i,j=1}^d a_{i,j}(x) \xi_i \xi_j \geq a_0 |\xi|^2.$$

The fractional derivative in time  $\partial^{\alpha}$  is understood in *Caputo* sense:

$$\partial^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t d\tau \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} & \text{if } m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t) & \text{if } \alpha = m. \end{cases}$$

Throughout this chapter, the initial conditions  $u_k(x)$  are bounded continuous (Hölder continuous, if  $d > 1$ ) functions. The study of the mild solution relies on the asymptote property of the Green's function  $Z, Y$  of the following deterministic equation.

$$\begin{cases} (\partial^{\alpha} - B) u(t, x) = f(t, x), & t > 0, x \in \mathbb{R}^d, \\ \left. \frac{\partial^k}{\partial t^k} u(t, x) \right|_{t=0} = u_k(x), & 0 \leq k \leq [\alpha] - 1, x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

In [3] we cover the case  $\alpha \in (1/2, 1)$ . When  $\alpha \in (1, 2)$ , [6] showed that when  $B$  is  $\Delta$ , Green's function  $Y$  of (1.2) the following:

$$Y(t, x) = C_d t^{\frac{\alpha}{2}(2-d)} f_{\frac{\alpha}{2}}(|x|t^{-\frac{\alpha}{2}}; d-1, \frac{\alpha}{2}(2-d)),$$

where

$$f_{\frac{\alpha}{2}}(z; \mu, \delta) = \begin{cases} \frac{2}{\Gamma(\frac{\mu}{2})} \int_1^{\infty} \phi(-\frac{\alpha}{2}, \delta; -zt)(t^2 - 1)^{\frac{\mu}{2}-1} dt, & \mu > 0, \\ \phi(-\frac{\alpha}{2}, \delta; -z), & \mu = 0; \end{cases}$$

$C_d = 2^{-n} \pi^{\frac{1-d}{2}}$  and the wright's function

$$\phi(-\frac{\alpha}{2}, \delta; -z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\delta - \frac{\alpha}{2}n)}$$

The solution of (1.2) has the following form:

$$u(t, x) = J_0(t, x) + \int_0^t ds \int_{\mathbb{R}^d} dy f(s, y) Y(t-s, x-y), \quad (1.3)$$

where and throughout the chapter, we denote

$$J_0(t, x) := \sum_{k=0}^{[\alpha]-1} \int_{\mathbb{R}^d} u_k(y) Z_{k+1}(t, x-y) dy. \quad (1.4)$$

For case of  $\alpha \in (1/2, 1)$ , we use  $Z$  in place of  $Z_1$ . We have the following facts about  $Z_1(t, x)$ ,  $Z_2(t, x)$  and  $Y(t, x)$ .

$$Z_1(t, x) = D^{\alpha-1} Y(t, x); \quad Z_1(t, x) = \frac{\partial}{\partial t} Z_2(t, x)$$

As in [3], We first get the estimation of  $Y$ , then use Wiener chaos expansion to obtain relation between the parameter  $\alpha, d, H_i$  and  $\kappa$  such that the mild solution exist.

The rest of the article is organized as follows. Section 2 gives more details about the solution of (1.1), estimation of  $Y$  for  $\alpha \in (1/2, 1)$  and some preliminaries about Wiener spaces. Section 3 gives the estimation of  $Y$  for  $\alpha \in (1, 2)$  and further estimations before proving the existence of the mild solution.

**Notation:** Throughout this chapter we denote

$$p(t, x) := \exp \left\{ -\sigma \left| \frac{x}{t^{\frac{\alpha}{2}}} \right|^{\frac{2}{2-\alpha}} \right\},$$

where  $\sigma > 0$  is a generic positive constant whose values may vary at different occurrence, so is  $C$ .

## 2 Preliminary

We consider a Gaussian noise  $W$  on a complete probability space  $(\Omega, \mathcal{F}, P)$  encoded by a centered Gaussian family  $\{W(\varphi); \varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)\}$ , whose covariance structure  $\lambda(s-t)$  is given by

$$\mathbb{E}(W(\varphi) W(\psi)) = \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(s, x) \psi(t, y) \lambda(s-t) \Lambda(x-y) dx dy ds dt, \quad (2.1)$$

where  $\lambda : \mathbb{R} \rightarrow \mathbb{R}_+$  is nonnegative definite and locally intergrable. Throughout the chapter, we denote

$$C_t := 2 \int_0^t \lambda(s) ds, \quad t > 0. \quad (2.2)$$

$\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is a fractional, Reisz or Bessel kernel.

**Definition 2.1.** Let  $Z$  and  $Y$  be the fundamental solutions defined by (1.2) and (1.3). An adapted random field  $\{u = u(t, x) : t \geq 0, x \in \mathbb{R}^d\}$  such that  $\mathbb{E}[u^2(t, x)] < +\infty$  for all  $(t, x)$  is a *mild solution* to (1.1), if for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ , the process

$$\{Y(t-s, x-y)u(s, y)1_{[0,t]}(s) : s \geq 0, y \in \mathbb{R}^d\}$$

is Skorodhod integrable (see (??)), and  $u$  satisfies

$$u(t, x) = J_0(t, x) + \int_0^t \int_{\mathbb{R}^d} Y(t-s, x-y)u(s, y)W(ds, dy) \quad (2.3)$$

almost surely for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ , where  $J_0(t, x)$  is defined by (1.4).

We use a similar chaos expansion to the one used in chapter 3. To prove the existence and uniqueness of the solution we show that for all  $(t, x)$ ,

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot, \cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 < \infty. \quad (2.4)$$

### 3 Estimations of the Green's functions

The fundamental solution of (1.2) is constructed by Levi's parametrix method. We refer the reader to [2] for detail of this method. In this section  $x := (x_1, x_2, \dots, x_d) \in \mathbb{R}^d, \xi, \eta$  are defined the same way;  $t \in (0, T]$ . We use  $\gamma$  to denote the Hölder exponents with respect to spatial variables. We can assume they are equal. For  $\alpha \in (1, 2)$ , we assume

$$\gamma > 2 - \frac{2}{\alpha}.$$

For  $\alpha \in (\frac{1}{2}, 1)$ , Chapter 4 gives the estimations the  $Z$  and  $Y$ . For  $\alpha \in (1, 2)$ , we need some lemmas before we can estimate  $Z_1, Z_2$  and  $Y$ .

From [5] we have

$$\begin{aligned} Z_j(t, x - \xi) &= Z_j^0(t, x - \xi, \xi) + V_{Z_j}(t, x; \xi), \quad j = 1, 2. \\ Y(t, x - \xi) &= Y_0(t, x - \xi, \xi) + V_Y(t, x; \xi). \end{aligned}$$

We refer the reader to [5] for the definitions of  $Z_k^0(t, x - \xi, \xi), Y_0(t, x - \xi, \xi)$  and  $V_Y(t, x; \xi)$ . Here we list their estimations which we use to get the estimations of  $Z_k$  and  $Y$  in section 3. These estimations are given in section 2.2 of [5] or Lemma 15 in [6].

**Lemma 3.1.**

$$\begin{aligned} |Z_1^0(t, x - \xi, \eta)| &\leq Ct^{-\frac{\alpha d}{2}} \mu_d(t^{-\frac{\alpha}{2}} |x - \xi|) p(t, x - \xi), \\ |Z_2^0(t, x - \xi, \eta)| &\leq Ct^{-\frac{\alpha d}{2} + 1} \mu_d(t^{-\frac{\alpha}{2}} |x - \xi|) p(t, x - \xi), \end{aligned}$$

where

$$\mu_d(z) := \begin{cases} 1, & d = 1; \\ 1 + |\log z|, & d = 2; \\ z^{2-d}, & d \geq 3. \end{cases} \quad (3.1)$$

**Lemma 3.2.**

$$|Y_0(t, x - \xi, \eta)| \leq Ct^{\alpha - \frac{\alpha d}{2} - 1} \mu_d(t^{-\frac{\alpha}{2}} |x - \xi|) p(t, x - \xi),$$

where

$$\mu_d(z) := \begin{cases} 1, & d \leq 3; \\ 1 + |\log z|, & d = 4; \\ z^{4-d}, & d \geq 5. \end{cases} \quad (3.2)$$

The following estimations of  $V_{Z_1}$ ,  $V_{Z_2}$  and  $V_Y$  are from Theorem 1 of [5], where  $\nu_1 \in (0, 1)$ , such that  $\gamma > \nu_1 > 2 - \frac{2}{\alpha}$  and  $\nu_0 = \nu_1 - 2 + \frac{2}{\alpha}$ .

**Lemma 3.3.**

$$|V_{Z_1}(t, x; \xi)| \leq \begin{cases} Ct^{(\gamma-1)\frac{\alpha}{2}} p(t, x - \xi), & d = 1; \\ Ct^{\nu_0\alpha-1} |x - \xi|^{-d+\gamma-\nu_1+2-\nu_0} p(t, x - \xi), & d \geq 2 \end{cases} \quad (3.3)$$

**Lemma 3.4.**

$$|V_{Z_2}(t, x; \xi)| \leq \begin{cases} Ct^{(\gamma-1)\frac{\alpha}{2}+1} p(t, x - \xi), & d = 1; \\ Ct^{\frac{\nu_0\alpha}{2}+1-\alpha} |x - \xi|^{-d+\gamma-\nu_1+2-\nu_0} p(t, x - \xi), & d \geq 2 \end{cases} \quad (3.4)$$

**Lemma 3.5.**

$$|V_Y(t, x; \xi)| \leq \begin{cases} Ct^{\alpha-1+(\gamma-1)\frac{\alpha}{2}} p(t, x - \xi), & d = 1; \\ Ct^{\nu_0\alpha-1} |x - \xi|^{-d+\gamma-\nu_1+2-\nu_0} p(t, x - \xi), & d \geq 2 \end{cases} \quad (3.5)$$

Based on the above three lemmas we have

**Lemma 3.6.** *Let  $x \in \mathbb{R}^d$ ,  $t \in (0, T]$ . Then*

$$|Y(t, x - \xi)| \leq \begin{cases} Ct^{-1+\frac{\alpha}{2}} p(t, x - \xi), & d = 1; \\ Ct^{\alpha-\frac{\alpha}{2}\gamma+\nu_0\alpha-2} |x - \xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x - \xi), & d \geq 2. \end{cases} \quad (3.6)$$

*Proof.* We "add" together the estimation of  $Y_0$  in Lemma 3.2 and  $V_y$  in Lemma 3.5 to get the estimation of  $Y$ . We use the following inequality throughout the proof.

$$a, b, \sigma > 0, \quad \text{then} \quad \exists \sigma', C > 0, \quad \text{s.t.} \quad x^a e^{-\sigma x^b} < C e^{-\sigma' x^b},$$

First when  $d = 1$ ,

$$\begin{aligned} |Y(t, x - \xi)| &\leq |Y_0(t, x - \xi, \xi)| + |V_Y(t, x, \xi)| \\ &\leq Ct^{\alpha-1+(\gamma-1)\frac{\alpha}{2}} p(t, x - \xi) + Ct^{-1+\frac{\alpha}{2}} p(t, x - \xi) \\ &\leq Ct^{-1+\frac{\alpha}{2}} p(t, x - \xi). \end{aligned}$$

When  $d \geq 5$ , by the fact

$$\nu_0 = \nu_1 - 2 + 2/\alpha \quad \text{and} \quad \gamma > \nu_1 > 2 - \frac{2}{\alpha},$$

we have

$$4 - \gamma + 2\nu_0 - \frac{2}{\alpha} = -\gamma + 2\nu_1 + \frac{2}{\alpha} \geq 0.$$

Therefore

$$\begin{aligned} |Y_0(t, x - \xi, \xi)| &\leq C t^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{4-d} p(t, x - \xi) \\ &= C t^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{4 - \gamma + 2\nu_0 - \frac{2}{\alpha}} p(t, x - \xi) \\ &\leq C t^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi). \end{aligned}$$

Furthermore because of the assumption

$$\gamma > 2 - \frac{2}{\alpha},$$

we have

$$\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2 < \nu_0\alpha - 1.$$

Therefore

$$\begin{aligned} |Y(t, x - \xi)| &\leq |Y_0(t, x - \xi, \xi)| + |V_y(t, x, \xi)| \\ &\leq C t^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi) \\ &\quad + C t^{\nu_0\alpha - 1} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi) \\ &\leq C t^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi). \end{aligned}$$

When  $d = 2$  and  $d = 3$ , as in the previous cases, we first have

$$\begin{aligned} |Y_0(t, x - \xi, \xi)| &\leq C t^{\alpha - \frac{\alpha n}{2} - 1} p(t, x - \xi) \\ &\leq C t^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi). \end{aligned}$$

Then as the last step in the case of  $d \geq 5$ , we have

$$|Y(t, x - \xi)| \leq C t^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi).$$

When  $d = 4$ , let's first transform the estimation of  $Y_0$  into the following form:

$$t^{\zeta_d} |x - \xi|^{\kappa_d} p(t, x - \xi).$$

We have

$$\begin{aligned} |Y_0(t, x - \xi, \xi)| &\leq C t^{\alpha - \frac{\alpha d}{2} - 1} \left\{ \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^\theta + \left| \frac{t^{\frac{\alpha}{2}}}{x - \xi} \right|^\theta \right\} p(t, x - \xi) \\ &\leq C t^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{t^{\frac{\alpha}{2}}}{x - \xi} \right|^\theta \left\{ \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{2\theta} + 1 \right\} p(t, x - \xi), \end{aligned}$$

for  $\forall \theta > 0$ .

If  $|\frac{x-\xi}{t^{\frac{\alpha}{2}}}| \leq 1$ , then

$$\left\{ \left| \frac{x-\xi}{t^{\frac{\alpha}{2}}} \right|^{2\theta} + 1 \right\} p(t, x - \xi) \leq 2p(t, x - \xi);$$

if  $|\frac{x-\xi}{t^{\frac{\alpha}{2}}}| > 1$ , then

$$\begin{aligned} \left\{ \left| \frac{x-\xi}{t^{\frac{\alpha}{2}}} \right|^{2\theta} + 1 \right\} p(t, x - \xi) &\leq 2 \left| \frac{x-\xi}{t^{\frac{\alpha}{2}}} \right|^{2\theta} p(t, x - \xi) \\ &\leq Cp(t, x - \xi). \end{aligned}$$

Therefore if we choose  $\theta > 0$  such that

$$-\theta > -d + \gamma - 2\nu_0 + \frac{2}{\alpha},$$

we have

$$\begin{aligned} |Y_0(t, x - \xi, \xi)| &\leq Ct^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{-\theta} p(t, x - \xi) \\ &\leq Ct^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi) \\ &\leq Ct^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi). \end{aligned}$$

As in previous two cases, we end up with

$$|Y(t, x - \xi)| \leq Ct^{\alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi).$$

□

Let's denote the the estimation function of  $Y$  by  $t^{\zeta_d} |x - \xi|^{\kappa_d} p(t, x - \xi)$ . For the estimation of integral (5.2) involving  $Y$  and fractional kernel it more convenient to represent the estimation of  $Y$  as the the product of one dimensional functions. To this purpose, as in the case of  $0 < \alpha < 1$ , the estimation of  $Y$  is represented as the product of one dimensional functions, which is shown in the following lemma.

**Lemma 3.7.** *Let  $x_i, \xi_i \in \mathbb{R}, t \in (0, T]$*

$$|Y(t, x - \xi)| \leq C \prod_{i=1}^d t^{\zeta_d/d} |x_i - \xi_i|^{\kappa_d/d} p(t, x_i - \xi_i), \quad (3.7)$$

where  $\zeta_d$  and  $\kappa_d$  are the powers of  $t$  and  $x - \xi$  in the estimation of  $Y$ , i.e.,

$$\zeta_d = \begin{cases} -1 + \frac{\alpha}{2}, & d = 1; \\ \alpha - \frac{\alpha}{2}\gamma + \nu_0\alpha - 2, & d \geq 2. \end{cases} \quad (3.8)$$

and

$$\kappa_d = \begin{cases} 0, & d = 1; \\ -d + \gamma - 2\nu_0 + \frac{2}{\alpha}, & d \geq 2. \end{cases} \quad (3.9)$$

**Lemma 3.8.**

$$\sup_{t,x} \left| \int_{\mathbb{R}^d} Z_{k+1}(t, x - \xi) u_k(t, \xi) d\xi \right| \leq C \quad k = 0, 1.$$

*Proof.* First recall that  $u_k(x)$  are bounded. Thanks to the following fact from [4]

$$\int_{\mathbb{R}^d} Z_1^0(t, x, \xi) d\xi = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} Z_2^0(t, x, \xi) d\xi = t,$$

we only need to show

$$\sup_{t,x} \int_{\mathbb{R}^d} V_{Z_j}(t, x, \xi) d\xi \leq C,$$

since  $u_k$  are bounded.

Let's consider the case  $d \geq 3$  and  $d = 2$  for  $V_{Z_1}$  as examples. When  $d \geq 3$ , by the estimation of  $V_{Z_1}$  in Lemma 3.1, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |V_{Z_1}(t, x, \xi)| d\xi &\leq \int_{\mathbb{R}^d} C t^{-\frac{\alpha d}{2}} \mu_d(t^{-\frac{\alpha}{2}} |x - \xi|) p(t, x - \xi) dy \\ &\leq \int_{\mathbb{R}^d} C t^{-\frac{\alpha d}{2} + d} \mu_d(z) p(t, x - \xi) dz \\ &\leq C t^{-\frac{\alpha d}{2} + d} \\ &\leq C, \end{aligned}$$

due to the fact  $t \in (0, T]$ .

For the case  $d = 2, Z_1$ , notice that

$$\forall \theta > 0, \exists C > 0 \quad \text{s.t.} \quad (\log |z| + 1) < c|z|^\theta,$$

as shown in the case of  $d=4$  in the proof of 3.6. Then the above argument ends proof. The proof for the rest of the cases is almost the same, so we omit it. □

## 4 Miscellaneous estimations

For fractional kernel, we need the following estimation, which is immediate from Corollary 15 of [3].

**Lemma 4.1.** *Let  $0 < r, s \leq T$  and*

$$2H_i + \frac{2\kappa_d}{d} > 0. \quad (4.1)$$



Then for any  $\rho_1, \tau_2 \in \mathbb{R}, \rho_1 \neq \tau_2$ , we have

$$\int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{2H_i-2} |\rho_2 - \rho_1|^{\frac{\kappa_d}{d}} |\tau_2 - \tau_1|^{\frac{\kappa_d}{d}} p(s, \rho_2 - \rho_1) p(r, \tau_2 - \tau_1) d\rho_1 d\tau_1 \leq C(s r)^{\theta_i},$$

where

$$\theta_i = \begin{cases} C(s r)^{\frac{H_i d + \kappa_d}{2d} \alpha}, & 2H_i - 2 + \kappa_d/d \neq -1; \\ C(s r)^{\frac{d\epsilon + \kappa_d + d}{4d} \alpha}, & 2H_i - 2 + \kappa_d/d = -1. \end{cases}$$

*Proof.* In Corollary 15 of [3], let  $\theta_1 = 2H_i - 2, \theta_2 = \kappa_d/d$ . Then notice that for  $0 < r \leq T$

$$\forall \epsilon < 0, \exists C > 0, \quad s.t. \quad \log r < C r^\epsilon.$$

□

The next lemma can be proved as in Lemma 11 of [3].

**Lemma 4.2.** *Let  $-1 < \beta \leq 0, x \in \mathbb{R}^d$ . Then, there is a constant  $C$ , dependent on  $\sigma, \alpha$  and  $\beta$ , but independent of  $\xi$  and  $s$  such that*

$$\int_{\mathbb{R}^d} |x|^\beta p(s, x - \xi) dx \leq C s^{\frac{\alpha\beta}{2} + \frac{\alpha}{2}d}.$$

For Bessel kernel, we need the following lemma.

**Lemma 4.3.** *Assume  $0 < s, r \leq T$  and  $y_1, y_2, z_1, z_2 \in \mathbb{R}^d$ , we have that*

$$\int_{\mathbb{R}^{2d}} |Y(r, y_1 - y_2) Y(s, z_1 - z_2)| \int_0^\infty \omega^{-\frac{\kappa}{2}-1} e^{-\omega} e^{\frac{-|y_1 - z_1|^2}{4\omega}} d\omega dy_1 dz_1 \leq C \cdot (r s)^\ell,$$

where

$$\ell := \zeta_d - \frac{\alpha}{4}\kappa + \frac{\alpha}{2}\kappa_d + \frac{\alpha}{2}d$$

*Proof.* Recall that the estimation of  $Y(t, x)$  in Lemma 9 of [3] and (3.6) has the following form:

$$C s^{\zeta_d} |x|^{\kappa_d} p(t, x).$$

By substituting  $Y$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} |Y(r, y_1 - y_2) Y(s, z_1 - z_2)| \int_0^\infty \omega^{-\frac{\kappa}{2}-1} e^{-\omega} e^{\frac{-|y_1 - z_1|^2}{4\omega}} d\omega dy_1 dz_1 \\ & \leq C \int_{\mathbb{R}^d} s^{\zeta_d} |z_2 - z_1^{\kappa_d}| p(s, z_2 - z_1) r^{\zeta_d} \int_0^\infty I \cdot \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega dz_1, \end{aligned}$$

where

$$I := \int_{\mathbb{R}^d} |y_2 - y_1|^{\kappa_d} \exp \left\{ -\sigma \left| \frac{y_2 - y_1}{r^{\frac{\alpha}{2}}} \right|^{\frac{2}{2-\alpha}} \right\} \exp \left\{ -\frac{|y_1 - z_1|^2}{4\omega} \right\} dy_1.$$

For  $I$ , we have two estimations:

$$\begin{aligned} I &\leq \int_{\mathbb{R}_d} |y_2 - y_1|^{\kappa_d} \exp \left\{ -\sigma \left| \frac{y_2 - y_1}{r^{\frac{\alpha}{2}}} \right|^{\frac{2}{2-\alpha}} \right\} dy_1 \\ &\leq Cr^{\frac{\alpha}{2}\kappa_d + \frac{\alpha}{2}d}, \end{aligned}$$

and

$$\begin{aligned} I &\leq \int_{\mathbb{R}_d} |y_2 - y_1|^{\kappa_d} \exp \left\{ -\frac{|y_1 - z_1|^2}{4\omega} \right\} dy_1 \\ &\leq C\omega^{\frac{\kappa_d}{2} + \frac{d}{2}}, \end{aligned}$$

by Lemma 4.2.

With the estimations of  $I$ , we have

$$\begin{aligned} \int_0^\infty I \cdot \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega &= \int_0^{r^\alpha} I \cdot \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega + \int_{r^\alpha}^\infty I \cdot \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega \\ &\leq r^{\frac{\alpha}{2}\kappa_d + \frac{\alpha}{2}d - \frac{\alpha}{2}\kappa} + \int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2} + \frac{d}{2}} \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega. \end{aligned}$$

For  $\int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2} + \frac{d}{2}} \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega$ ,

if  $\frac{\kappa_d}{2} + \frac{d}{2} - \frac{\kappa}{2} < 0$

$$\begin{aligned} \int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2} + \frac{d}{2}} \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega &\leq \int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2} + \frac{d}{2} - \frac{\kappa}{2} - 1} d\omega \\ &= Cr^{\alpha(\frac{\kappa_d}{2} + \frac{d}{2} - \frac{\kappa}{2})}; \end{aligned}$$

if  $\frac{\kappa_d}{2} + \frac{d}{2} - \frac{\kappa}{2} \geq 0$

$$\begin{aligned} \int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2} + \frac{d}{2}} \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega &= \int_{r^\alpha}^\infty \omega^{\frac{\kappa_d}{2} + \frac{d}{2} - \frac{\kappa}{2} - 1} e^{-\omega} d\omega \\ &= C \\ &\leq Cr^{\alpha(\frac{\kappa_d}{2} + \frac{d}{2} - \frac{\kappa}{2})}. \end{aligned}$$

Therefore we end up with

$$\int_0^\infty I \cdot \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega \leq Cr^{\alpha(\frac{\kappa_d}{2} + \frac{d}{2} - \frac{\kappa}{2})}.$$

The estimation of integration with respect to  $z_1$  is straightforward thank to fact that  $C$  is independent of  $z_1$ .

We have

$$\int_{\mathbb{R}^d} s^{\zeta_d} |z_2 - z_1| p(s, z_2 - z_1) r^{\zeta_d} \int_0^\infty I \cdot \omega^{-\frac{\kappa}{2}-1} e^{-\omega} d\omega dz_1$$

$$\leq C r^{\alpha(\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2})} \cdot r^{\zeta_d} \cdot s^{\alpha(\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2})} s^{\zeta_d},$$

by Lemma 4.2.

By symmetry, we have

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} |Y(r, y_1 - y_2) Y(s, z_1 - z_2)| \int_0^\infty \omega^{-\frac{\kappa}{2}-1} e^{-\omega} e^{\frac{-|y_1 - z_1|^2}{4\omega}} d\omega dy_1 dz_1 \\ & \leq C s^{\alpha(\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2})} \cdot s^{\zeta_d} \cdot r^{\alpha(\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2})} r^{\zeta_d}. \end{aligned}$$

Combining the two estimations we get the estimation in the lemma.  $\square$

The following lemma is Theorem 3.5 from [1].

**Lemma 4.4.** *Let  $T_n(t) = \{s = (s_1, \dots, s_n) : 0 < s_1 < s_2 < \dots < s_n < t\}$ . Then*

$$\int_{T_n(t)} [(t - s_n)(s_n - s_{n-1}) \dots (s_2 - s_1)]^h ds = \frac{\Gamma(1+h)^n}{\Gamma(n(1+h) + 1)} t^{n(1+h)},$$

*if and only if  $1+h > 0$ .*

## 5 Existence and uniqueness of the solution

**Theorem 5.1.** *Assume the following conditions:*

- (1)  $\lambda(t)$  is a nonnegative definite locally integrable function;
- (2)  $\alpha \in (1/2, 1) \cup (1, 2)$ .

*Then relation (2.4) holds for each  $(t, x)$ , if any of the following is true. Consequently, equation (1.1) admits a unique mild solution in the sense of Definition 2.1.*

(i)  $\Lambda(x)$  is fractional kernel with condition:

$$H_i > \begin{cases} \frac{1}{2}, & d = 1, 2, 3, 4 \\ 1 - \frac{2}{d} - \frac{\gamma}{2d}, & d \geq 5, \alpha \in (0, 1) \\ 1 - \frac{2}{d}, & d \geq 5, \alpha \in (1, 2) \end{cases}$$

and

$$\sum_{i=1}^d H_i > d - 2 + \frac{1}{\alpha}.$$

(ii)  $\Lambda(x)$  is the Reisz or Bessel kernel with condition:

$$\kappa < 4 - 2/\alpha;$$

*Proof.* Fix  $t > 0$  and  $x \in \mathbb{R}^d$ .

Let

$$\begin{aligned}(s, y, t, x) &:= (s_1, y_1, \dots, s_n, y_n, t, x); \\ g_n(s, y, t, x) &:= \frac{1}{n!} Y(t - s_{\sigma(n)}, x - y_{\sigma(n)}) \cdots Y(s_{\sigma(2)} - s_{\sigma(1)}, y_{\sigma(2)} - y_{\sigma(1)}); \\ f_n(s, y, t, x) &:= g_n(s, y, t, x) J_0(s_{\sigma(1)}, x_{\sigma(1)}),\end{aligned}$$

where  $\sigma$  denotes a permutation of  $\{1, 2, \dots, n\}$  such that  $0 < s_{\sigma(1)} < \dots < s_{\sigma(n)} < t$ .

By iteration of  $u(t, x)$ , we have

$$\begin{aligned}n! \|f_n(\cdot, \cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \\ = n! \int_{[0, t]^{2n}} ds dr \int_{\mathbb{R}^{2nd}} dy dz f_n(s, y, t, x) f_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \prod_{i=1}^n \lambda(s_i - r_i),\end{aligned} \quad (5.1)$$

where  $dy := dy_1 \cdots dy_n$ , the differentials  $dz$ ,  $ds$  and  $dr$  are defined similarly. Set  $\mu(d\xi) := \prod_{i=1}^n \mu(d\xi_i)$ .

Recall that  $J_0$  is bounded, so we have

$$\begin{aligned}n! \|f_n(\cdot, \cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \\ \leq C \frac{1}{n!} \int_{[0, t]^{2n}} ds dr \int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \prod_{i=1}^n \lambda(s_i - r_i).\end{aligned}$$

Furthermore by Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \\ \leq \left\{ \int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(s, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \right\}^{1/2} \\ \cdot \left\{ \int_{\mathbb{R}^{2nd}} dy dz g_n(r, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \right\}^{1/2}\end{aligned}$$

(i) Let  $\Lambda(\cdot) = \varphi_H(\cdot)$  and use the estimation of  $Y$  in Lemma 3.6, we have

$$\begin{aligned}\int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(s, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \\ \leq C \prod_{i=1}^d \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_i dz_i\end{aligned} \quad (5.2)$$

where

$$\Theta_n(t, y_{ik}, s) := |s_{\sigma(k+1)} - s_{\sigma(k)}|^{\frac{\zeta_d}{d}} |y_{i\sigma(k+1)} - y_{i\sigma(k)}|^{\frac{\kappa_d}{d}} p(s_{\sigma(k+1)} - s_{\sigma(k)}, y_{i\sigma(k+1)} - y_{i\sigma(k)});$$

$$y_i = (y_{i1}, y_{i2}, \dots, y_{ik}, \dots, y_{in}), \quad z_i = (z_{i1}, z_{i2}, \dots, z_{ik}, \dots, z_{in});$$

$$dy_i := \prod_{k=1}^n dy_{ik} \quad dz_i := \prod_{k=1}^n dz_{ik};$$

and

$$y_{\sigma(k+1)} = z_{\sigma(k+1)} := x_i; \quad s_{\sigma(n+1)} = r_{\sigma(n+1)} := t.$$

Let's first consider the case  $2H_i - 2 + \kappa_d/d \neq -1$ . Applying Lemma 4.1 to

$$\int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_i dz_i \quad (5.3)$$

for  $dy_{i\sigma(1)} dz_{i\sigma(1)}$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_i dz_i \\ & \leq C(s_{i\sigma(2)} - s_{i\sigma(1)})^{2\ell_i} \int_{\mathbb{R}^{2n}} \prod_{k=2}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_i dz_i \end{aligned}$$

where

$$\ell_i = \frac{\zeta_d}{d} + \theta_i.$$

Applying Lemma 4.1 to (5.3) for  $dy_{i\sigma(k)} dz_{i\sigma(k)}, k = 2, \dots, n$ , we have

$$\prod_{i=1}^d \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_i dz_i \leq \prod_{k=1}^n C^m (s_{\sigma(k+1)} - s_{\sigma(k)})^{2\ell}$$

where

$$\ell = \sum_{i=1}^d \ell_i = \zeta_d + \frac{|H|\alpha}{2} + \frac{\kappa_d \alpha}{2} \quad \text{with} \quad |H| = \sum_{i=1}^d H_i. \quad (5.4)$$

Due to the same argument, we have

$$\prod_{i=1}^d \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, r) \Theta_n(t, z_{ik}, r) dy_i dz_i \leq \prod_{k=1}^n C^m (r_{\rho(k+1)} - r_{\rho(k)})^{2\ell}$$

Therefore

$$\int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \leq C^m (\phi(s)\phi(r))^\ell,$$

where

$$\phi(s) := \prod_{i=1}^n (s_{\sigma(i+1)} - s_{\sigma(i)}), \quad \phi(r) := \prod_{i=1}^n (r_{\rho(i+1)} - r_{\rho(i)}),$$

with

$$0 < s_{\sigma(1)} < s_{\sigma(2)} < \dots < s_{\sigma(n)} \quad \text{and} \quad 0 < r_{\rho(1)} < r_{\rho(2)} < \dots < r_{\rho(n)}.$$

Hence,

$$\begin{aligned} n! \|f_n(\cdot, \cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 &\leq \frac{C^n}{n!} \int_{[0,t]^{2n}} \prod_{i=1}^n \lambda(s_i - r_i) (\phi(s)\phi(r))^\ell ds dr \\ &\leq \frac{C^n}{n!} \frac{1}{2} \int_{[0,t]^{2n}} \prod_{i=1}^n \lambda(s_i - r_i) (\phi(s)^{2\ell} + \phi(r)^{2\ell}) ds dr \\ &= \frac{C^n}{n!} \int_{[0,t]^{2n}} \prod_{i=1}^n \lambda(s_i - r_i) \phi(s)^{2\ell} ds dr \\ &\leq \frac{C^n C_t^n}{n!} \int_{[0,t]^n} \phi(s)^{2\ell} ds \\ &= C^n C_t^n \int_{T_n(t)} \phi(s)^{2\ell} ds \\ &= \frac{C^n C_t^n \Gamma(2\ell + 1)^n t^{(2\ell+1)n}}{\Gamma((2\ell + 1)n + 1)}, \end{aligned}$$

where  $C_t = 2 \int_0^t \lambda(r) dr$ . The last step is by Lemma 4.4.

Therefore,

$$n! \|f_n(\cdot, \cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \leq \frac{C^n C_t^n}{\Gamma((2\ell + 1)n + 1)},$$

and  $\sum_{n \geq 0} n! \|f_n(\cdot, \cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2$  converges if  $\ell > -1/2$ .

Next we need to show

$$\ell > -1/2 \iff |H| > d - 2 + \frac{1}{\alpha}.$$

Firstly by definition of  $\ell$ , (5.4)

$$\ell > -1/2 \iff |H| > -\frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha} \zeta_d.$$

Then using the definition of  $\zeta_d$  and  $\kappa_d$  in (4.2), (4.3) of [3] for  $1/2 < \alpha < 1$  and (3.8), (3.9) for  $1 < \alpha < 2$ , we have:

when  $1/2 < \alpha < 1$ ,

$$\frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha} \zeta_d = \begin{cases} -1 + \frac{1}{\alpha}, & d = 1; \\ \frac{1}{\alpha}, & d = 2; \\ \frac{1}{\alpha} + 2, & d = 4; \\ \frac{1}{\alpha} - 2 + d, & d = 3 \text{ or } d \geq 5; \end{cases}$$

when  $1 < \alpha < 2$ ,

$$\frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha}\zeta_d = \begin{cases} -1 + \frac{1}{\alpha}, & d = 1; \\ d - 2 + \frac{1}{\alpha}, & d \geq 2; \end{cases}$$

For case  $2H_i - 2 + \kappa_d/d = -1$ , applying Lemma 4.1 to (5.3), we have

$$\prod_{i=1}^d \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_i dz_i \leq \prod_{k=1}^n C^n (s_{\sigma(k+1)} - s_{\sigma(k)})^{2\ell'},$$

where

$$\ell' = \zeta_d + \frac{d\epsilon + \kappa_d + d}{4}\alpha \quad \text{with} \quad |H| = \sum_{i=1}^d H_i.$$

Using the relation  $2H_i - 2 + \kappa_d/d = -1$ , we have

$$\ell' = \ell + \frac{d\alpha}{4}\epsilon.$$

Since

$$|H| > d - 2 + \frac{1}{\alpha} \implies \ell > -1/2,$$

we can choose  $\epsilon$  big enough such such

$$|H| > d - 2 + \frac{1}{\alpha} \implies \ell' > -1/2.$$

Lastly, when  $\alpha \in (1/2, 1)$ , for  $d \leq 4$ ,  $H_i > 1/2$  implies condition (4.1); for  $d > 4$ , condition (4.1) is implied by

$$H_i > 1 - \frac{2}{d} - \frac{\gamma}{2d}$$

with  $\gamma_0$  sufficiently small; when  $\alpha \in (1, 2)$  for  $d = 1$ ,  $H_i > 1/2$  implies (4.1); for  $d \geq 2$ , (4.1) is implied by

$$H_i > 1 - \frac{2}{d}$$

with  $\nu_0$  sufficiently small. This completes the proof of Theorem 5.1 for case of  $\Lambda(\cdot) = \varphi_H(\cdot)$

(ii) Let  $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ . For Reisz kernel, notice that

$$|x|^{-\kappa} \leq C \prod_{i=1}^d |x_i|^{\frac{\kappa}{d}},$$

so this case is reduced to case (i) with  $H_i = (-\frac{\kappa}{d} + 2)\frac{1}{2}$ ,  $i = 1, 2, \dots, d$ .

Correspondingly

$$|H| > d - 2 + \frac{1}{\alpha}$$

is

$$\kappa < 4 - 2/\alpha,$$

which also guarantees condition (4.1) .

For Bessel kernel, applying Lemma 4.3 for  $dy_{\sigma(i)}dz_{\sigma(i)}$  in the order of  $i = 1, 2, \dots, n$  to

$$\int_{\mathbb{R}^{2nd}} dydz g_n(s, y, t, x)g_n(s, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i)$$

yields

$$\int_{\mathbb{R}^{2nd}} dydz g_n(s, y, t, x)g_n(s, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \leq \prod_{k=1}^n C^n(s_{\sigma(k+1)} - s_{\sigma(k)})^{2\ell},$$

where

$$\ell := \zeta_d - \frac{\alpha}{4}\kappa + \frac{\alpha}{2}\kappa_d + \frac{\alpha}{2}d$$

As in case (i),  $\sum_{n \geq 0} n! \|f_n(\cdot, \cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2$  converges if  $\ell > -1/2$ . Then using the definition of  $\zeta_d$  and  $\kappa_d$  in (4.2), (4.3) of [3] for  $1/2 < \alpha < 1$  and (3.8), (3.9) for  $1 < \alpha < 2$ , we have

$$\ell > -1/2 \iff \kappa < 4 - 2/\alpha.$$

This finishes the the proof of the theorem. □

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